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TWO THEOREMS ON ANTICANONICAL MODELS OF RATIONAL SURFACES

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Let X be a non-singular rational surface over an algebraically closed field k. If K is a canonical divisor of X, then the divisor -K is called an anticanonical divisor of X. The anti-Kodaira dimension(or the anticanonical dimension) $\kappa^{-1}(X)$ of X is defined to be $\kappa(-K,X)$. In this note, we state two theorems on the structure of a rational surface X with $\kappa^{-1}(X)=2$. Details will be discussed elsewhere.

Notation:

$$R^{-1}(X)=\bigoplus_{m\geq 0} H^0(X,O(-mK)) \qquad \text{(the anticanonical ring)}$$

$$P_{-m}(X)=\dim H^0(X,O(-mK)) \qquad \text{(the m-th antigenus, m>0)}$$

Theorem I. Let X be a non-singular rational surface with $\kappa^{-1}(X)=2$. Then $R^{-1}(X)$ is finitely generated and the anticanonical model Y (= Proj $R^{-1}(X)$) of X satisfies

(i) Y has only isolated rational singularities,

(ii) some multiple of $-K_Y$ is an ample Cartier divisor.

If X contains no redundant exceptional curves(For the definition, see §2), then X coincides with the minimal resolution of Y.

Conversely, if a normal projective surface Y satisfies (i), (ii), then the minimal resolution X of Y is a rational surface with $\kappa^{-1}(X)=2$.

Corollary of Proof. Let X be a non-singular rational surface with $\kappa^{-1}(X)=2$. Then the number of irreducible curves with negative self-intersection on X is finite.

Theorem II (char(k)=0). Let X be a non-singular rational surface with $\kappa^{-1}(X)=2$ (without redundant exceptional curves). By Theorem I, X is the minimal resolution of its anticanonical model Y, $\pi:X\longrightarrow Y$. We have the following dimension formula (for $m\geqq 0$):

$$P_{-m}(X)=\frac{(m+1)m}{2}K^2+1+\sum_{y_i\in \mathrm{Sing}\, Y}\ell_m(y_i),$$

where the last term is given by

$$\ell_m(y_i)=\dim R^1\pi_*O(-mK)_{y_i}.$$

There are many distinguished rational surfaces belonging to the class $\kappa^{-1}=2$ such as:(1) $\mathbf{P}^2$, $\mathbf{F}_e$, (2) Del Pezzo surfaces, (3) rational surfaces obtained by torus embeddings, (4) minimal normal compactifications of $\mathbf{C}^2$, etc.

§1. Surface Singularities

We collect some facts concerning surface singularities. Let V be an affine surface having only one normal singularity y. We denote by $\pi:U\longrightarrow V$ the minimal resolution of y in the sense that there exist no exceptional curves of the first kind over y. Set

$$A=\pi^{-1}(y)=E_1\cup\cdots\cup E_k.$$

Let K denote a canonical divisor of U.

(a) Let K_V denote a canonical divisor (as a Weil divisor) of V. If $i : V\setminus y \hookrightarrow V$ is the inclusion map, then

$$O(-mK_V) \cong i_* O(-mK_{V\setminus y}).$$

Lemma 1. $\pi_* O(-mK) \cong O(-mK_V)$ for $m>0$.

Proof. We have an exact sequence

$$0 \longrightarrow H^0(U, O(-mK)) \longrightarrow H^0(U\setminus A, O(-mK)) \longrightarrow H^1_A(U, O(-mK)).$$

By duality,[1] $H^1_A(U.O(-mK)) \cong H^1(U, O((m+1)K))^\vee$, which vanishes under the hypothesis that π is minimal (See Appendix). Thus

$$H^0(U, O(-mK)) \cong H^0(U\setminus A, O(-mK)) \cong H^0(V\setminus y, O(-mK_{V\setminus y})) \cong H^0(V, O(-mK_V)).$$

Q.E.D.

Corollary. Let Y be a normal projective surface. Let X be the minimal resolution of Y, Then

$$R^{-1}(X) \cong R^{-1}(Y),$$

where we defined naturally as: $R^{-1}(Y) = \bigoplus_{m \geqq 0} H^0(Y, O(-mK_Y))$.

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1) In general, for a divisor D on U, $H^1_A(U, O(D)) \cong H^1(U, O(K-D))^\vee$. For a proof, see [2]. See also Lipman : Ann. of Math. 107(1978). In the complex category, one can instead understand as follows(cf. Laufer : Amer.J.Math.94(1972)). There is an exact sequence

$$0 \longrightarrow H^0(U, O(D)) \longrightarrow H^0_\infty(U, O(D)) \longrightarrow H^1_c(U, O(D)),$$

where H^1_c denotes the cohomology with compact support. It is known that

$$H^0_\infty(U, O(D)) \cong H^0(U\setminus A, O(D)).$$

So the usual Serre duality $H^1_c(U, O(D)) \cong H^1(U, O(K-D))^\vee$ can be used.

(b) Define a Q-divisor $\Delta=\Sigma\delta_i E_i$ by the equations:

$$\Sigma\delta_i E_i E_j = -KE_j \quad \text{for } j=1,\ldots,k.$$

We have $\Delta \geqq 0$. Note that $\Delta=0$ if and only if y is a rational double point and otherwise $\mathrm{Supp}(\Delta)=A$.

Remark. It is known that $[\Delta]=0$ if and only if y is a quotient singularity for the case in which $k=\mathbb{C}$ (Watanabe,K.: Math.Ann.250(1980)). If y is a Gorenstein singularity, then Δ is integral and we have

$$\pi^*O(K_V)\cong O(K+\Delta).$$

In general, we have the isomorphism:

$$O(mK_V)\cong\pi_*O(mK+[m\Delta]) \qquad \text{for } m\geqq 0.$$

This can be shown in a similar manner to that in Lemma 1, using a vanishing theorem(Remark in Appendix). Here [ ] denotes the integral part.

(c) By definition, the singularity y is rational if $R^1\pi_*O_U=0$.

Characterization([1]): The singularity y is rational if and only if $H^1(Z,O_Z)=0$ for every effective divisor Z supported in A (cf.Appendix).

Lemma 2. Suppose that y is a rational singularity. Let r be the least integer such that $\tilde{\Delta}=r\Delta$ is integral. Then the divisor rK_V is a Cartier divisor and we have

$$\pi^*(rK_V)\sim rK+\tilde{\Delta}.$$

Proof. This fact is more or less known.

§2. Proof of Theorem I

Let X be a non-singular rational surface with $\kappa^{-1}(X)=2$. There is a unique Zariski decomposition of -K (cf.[3], [7], see also [6]).

$$-K=P+N$$

where the P is a numerically effective Q-divisor and the N is an effective Q-divisor satisfying

(i) N=0 or the intersection matrix of the irreducible components is negative definite,

(ii) the intersection of P with each irreducible component of N is zero.

We know that

$$\kappa^{-1}(X)=2 \iff P^2>0.$$

We study the set

$$A=\{\text{irreducible curves } E \mid PE=0\}.$$

The intersection matrix of irreducible components of A is negative definite(Hodge Index Theorem). So A is a finite set. We note that $\mathrm{Supp}(N)\subset A$.

Step 1. An exceptional curve of the first kind in A is said to be redundant.

Claim. We may assume that X contains no redundant exceptional curves.

Suppose that E is a redundant exceptional curve. Let $\mu:X\longrightarrow X'$ be the contraction of E. Also $\kappa^{-1}(X')=2$. It can be shown that $R^{-1}(X)\cong R^{-1}(X')$. So by successive such contractions, we may assume from the first that X contains no redundant exceptional curves.

Step 2. We decompose N into connected components:

$$N=N_1+\cdots+N_s.$$

Put $A_i=\mathrm{Supp}(N_i)$.

Claim. Each A_i can be contracted to a rational singularity.

Proof. Let Z be an effective divisor supported in A_i. Since $K+Z\sim -P-N+Z$, we get $(K+Z)P<0$. It follows that $\kappa(K+Z,X)=-\infty$. Thus

$$0=H^0(X,\mathcal{O}(K+Z))\longrightarrow H^0(Z,\omega_Z)\longrightarrow H^1(X,\mathcal{O}(K))=0.$$

Hence we get $H^0(Z,\omega_Z)=0$. By duality, we obtain $H^1(Z,\mathcal{O}_Z)=0$. Q.E.D.

Step 3. Let E be an irreducible curve in $A\setminus\mathrm{Supp}(N)$. We can easily see that E is a non-singular rational curve satisfying $E^2=-2$, $NE=0$. So E is disjoint from $\mathrm{Supp}(N)$. Now let $A_{s+1},\ldots, A_{s+t}$ be connected components of $A\setminus\mathrm{Supp}(N)$. We find that each A_{s+j} can be contracted to a rational double point.

As a consequence, we have a contraction $\pi:X\longrightarrow Y$ of A to a normal projective surface Y (cf.[1]). Let y_i be the singularity corresponding to A_i. Since A contains no exceptional curves of the first kind, π is nothing but the minimal resolution of Y.

Step 4. We denote by Δ_i the numerically anticanonical divisor of y_i (as defined in §1, (b)). Then the divisor N_i coincides with Δ_i. We fix the notation. Let r_i be the least integer such that $r_i\Delta_i$ is integral. Put $r=\mathrm{l.c.m.}(r_i)$. By Lemma 2, rK_Y is a Cartier divisor. Also we get

$$\pi^*(-rK_Y)\sim rP.$$

Claim. $-rK_Y$ is an ample Cartier divisor.

Cosequently, $R^{-1}(Y)$ is a finitely generated graded ring and we have $Y=\mathrm{Proj}\ R^{-1}(Y)$. Since we have seen $R^{-1}(X)\cong R^{-1}(Y)$, we conclude that Y is the anticanonical model of $\dot{X}$.

To prove the coverse implication, we first show that $\kappa^{-1}(X)=2$. So X is a ruled surface. By looking at the relatively minimal model of X, we can prove that X is rational. Q.E.D.

§3. Proof of Theorem II

By the Riemann-Roch theorem, we have

$$P_{-m}(X)=\frac{(m+1)m}{2}K^2+1+\dim H^1(X,O(-mK)) \qquad \text{for } m\geq 0.$$

In order to prove Theorem II, it suffices therefore to see the last term. By the Leray sequence (together with Lemma 1), we have

$$H^1(Y,O(-mK_Y))\longrightarrow H^1(X,O(-mK))\longrightarrow H^0(Y,R^1\pi_*O(-mK))\rightarrow H^2(Y,O(-mK_Y)).$$

By definition

$$\dim H^0(Y,R^1\pi_*O(-mK))=\Sigma\ell_m(y_i).$$

We are reduced to prove the following

Claim. $H^i(Y,O(-mK_Y))=0$ for $i>0$, $m\geq 0$.

We restrict ourselves to the case in which char(k)=0. We need

Miyaoka-Ramanujam Vanishing Theorem([4]). Let D be a divisor on a non-singular projecitve surface X. Suppose that $\kappa(D,X)=2$. Let D=P+N denote the Zariski decomposition. Then

$$H^i(X,O(K+D-[N]))=0 \qquad \text{for } i>0.$$

We apply this to the divisor -(m+1)K. Then we get

$$H^i(X,O(-mK-[(m+1)N]))=0 \qquad \text{for } i>0.$$

Put $G=[(m+1)N]$. We have a local version of the above vanishing theorem (For the formulation, see Appendix), which proves

$$R^1\pi_*O(-mK-G)=0.$$ [2)]

It follows that

$$H^1(Y,\pi_*O(-mK-G))\cong H^1(X,O(-mK-G))=0.$$

We have an exact sequence

$$0\longrightarrow\pi_*O(-mK-G)\longrightarrow\pi_*O(-mK)\longrightarrow\mathcal{J}\longrightarrow 0,$$

where the sheaf $\mathcal{J}$ is a torsion sheaf supported in Sing(Y). By using the cohomology sequence, we obtain the required result. Q.E.D.

§ 4. Examples

We give two simple examples. We define the degree of X by (with the notation in §2)

$$d(X)=P^2.$$

(1) Take a nodal cubic curve C in $\mathbb{P}^2$. Let $\mu:X\dashrightarrow\mathbb{P}^2$ be the blowing up of the node n-times. Let E_0 be the strict transform of C and let $E_1,\ldots,E_n$ be the strict transform of the exceptional curves. We have

$$E_0^2=-(6-n),\ E_1^2=\cdots=E_{n-1}^2=-2,\ E_n^2=-1.$$

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2) In our case, as is pointed out by S.Mori, this follows from the global vanishing theorem. Take an ample invertible sheaf $\mathcal{L}$ on Y. By using the Miyaoka-Ramanujam vanishing theorem, we get

$$R^1\pi_*O(-mK-G)\otimes\mathcal{L}^{\otimes n}=0$$

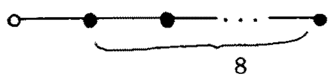
for a large integer n. We infer from this that $R^1\pi_*O(-mK-G)=0$.

These curves form a cycle of $\mathbb{P}^1$'s. In this case, we have $\kappa^{-1}(X)=2$. For instance, for n=9, we have

$$P=\frac{10}{19}E_0+\frac{11}{19}E_1+\cdots+\frac{18}{19}E_8+E_9$$
$$N=\frac{9}{19}E_0+\frac{8}{19}E_1+\cdots+\frac{1}{19}E_8$$

$$P_{-1}(X)=1, \qquad d(X)=\frac{9}{19}.$$

The anticanonical model Y has one rational triple point with the dual graph



where ● (resp. ○)denotes a non-singular rational curve of self-intersection -2 (resp. -3).

(2) Consider an action of $G=\mathbb{Z}/5\mathbb{Z}$ on $\mathbb{P}^2$ (over $\mathbb{C}$) by

$$(X_0:X_1:X_2)\longrightarrow(X_0:\zeta X_1:\zeta^2X_2)$$

where the ζ is a primitive 5-th root of 1. Then G has three fixed points. The quotient $Y=\mathbb{P}^2/G$ has two rational triple points and a rational double point.

Let X be the minimal resolution of Y. Then X is a rational surface with $\kappa^{-1}(X)=2$ and Y is the anticanonical model of X. In this case

$$P_{-1}(X)=2, \qquad d(X)=\frac{9}{5}.$$

In general, if a finite group G acts freely except a finite number of points on a non-singular rational surface X' with $\kappa^{-1}(X')=2$, the minimal resolution X of the quotient X'/G is again a rational surface with $\kappa^{-1}(X)=2$. Furthermore $d(X)=d(X')/|G|$.

Appendix. Local Vanishing Theorem

Let V, y, π, U, $E_1,\ldots,E_k$ have the same meaning as in §1. But we do not assume that π is minimal. Given a divisor D on U, there exists a unique Zariski decomposition (local version):

$$D=P+N,$$

where the P is a Q-divisor such that $PE_j\geqq 0$ for $j=1,\ldots,k$ and the N is an effective Q-divisor supported in A with the property that $PE_j=0$ for each E_j contained in N.

Theorem A. We have

$$R^1\pi_*O(K+D-[N])=0.$$

Remark. When $N=0$, this is the Laufer-Ramanujam vanishing theorem. When π is minimal, the Zariski decompositions of K and $-K$ are as follows

	P	N
K	K	0
-K	$-K-\Delta$	Δ

We obtain therefore two vanishings:

$$R^1\pi_*O(mK)=0 \qquad \text{for } m>0 \quad (\text{cf. [2]}),$$

$$R^1\pi_*O(-mK-[(m+1)\Delta])=0 \qquad \text{for } m\geqq 0.$$

As a corollary, we get the formula:

$$\ell_m(y)=\dim R^1\pi_*\,O(-mK)=\dim H^1([(m+1)\Delta],\ O_{[(m+1)\Delta]}(-mK)).$$

In particular, we have

$$\dim R^1\pi_*O_U=\dim H^1([\Delta],O_{[\Delta]}).$$

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